

# Willmore Surfaces of Constant Möbius Curvature

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## Abstract

We study Willmore surfaces of constant Möbius curvature  $\mathcal{K}$  in  $\mathbb{S}^4$ . It is proved that such a surface in  $\mathbb{S}^3$  must be part of a minimal surface in  $\mathbb{R}^3$  or the Clifford torus. Another result in this paper is that an isotropic surface (hence also Willmore) in  $\mathbb{S}^4$  of constant  $\mathcal{K}$  could only be part of a complex curve in  $\mathbb{C}^2 \cong \mathbb{R}^4$  or the Veronese 2-sphere in  $\mathbb{S}^4$ . It is conjectured that they are the only examples possible. The main ingredients of the proofs are over-determined systems and isoparametric functions.

## 1 Introduction

In this paper we consider Willmore surfaces in  $\mathbb{S}^4$  of constant Möbius curvature.

Our interest in such surfaces originates from the analogous problem about minimal surfaces in  $\mathbb{S}^n$ . It was well known that minimal surfaces are quite rigid objects. In particular, minimal surfaces in  $\mathbb{S}^n$  whose induced metric are of constant Gaussian curvature  $K$  can be determined even locally (see [4] and references therein). Such a surface is always homogeneous, and locally it must be congruent to a generalized Veronese 2-sphere (whose curvature is  $2/m(m+1)$  for some integer  $m$ ) when  $K > 0$ , or to a generalized Clifford surface when  $K = 0$  (c.f. [7, 15] for construction of these examples); the case  $K < 0$  is impossible. This remarkable result has been generalized to minimal surfaces in real space forms  $\mathbb{R}^n, \mathbb{H}^n$  [4] or in complex space form  $\mathbb{CP}^n$  (see [17] and references therein).

In Möbius geometry, associated with any surface without umbilic points there is a conformally invariant metric (induced from the conformal Gauss map [3]) called the Möbius metric, whose curvature is named the Möbius curvature, and denoted by  $\mathcal{K}$ . The area of this surface under the Möbius metric is exactly the well-known Willmore functional. Thus Willmore surfaces, the critical surfaces with respect to Willmore functional, are also called Möbius minimal surfaces. More interestingly, minimal surfaces in space forms  $\mathbb{S}^n, \mathbb{R}^n, \mathbb{H}^n$  are themselves also Willmore surfaces [2, 31]. This raises the natural problem that whether we can generalize the remarkable classification result above to Willmore surfaces of constant Möbius curvature.

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In 3-space the only known examples of such surfaces are minimal surfaces in  $\mathbb{R}^3$  (with  $\mathcal{K} \equiv 1$ ) and the Clifford torus (with  $\mathcal{K} \equiv 0$ ). In 4-space we might add complex curves in  $\mathbb{C}^2 \cong \mathbb{R}^4$  (with  $\mathcal{K} \equiv 2$ ) and the Veronese sphere (with  $\mathcal{K} \equiv \frac{1}{2}$ ). Naturally we have the following

**Conjecture.** *Any Willmore surface of constant Möbius curvature is locally Möbius equivalent to one of the four examples mentioned above (minimal surfaces in  $\mathbb{R}^3$ , Clifford torus, complex curve in  $\mathbb{C}^2$ , and Veronese sphere).*

We can confirm this conjecture under various additional assumptions. First comes the codim-1 case:

**Theorem A.** *Let  $y : M \rightarrow \mathbb{S}^3$  be a Willmore surface without umbilic points and  $\mathcal{K} = \text{constant}$ . Then locally  $y$  is Möbius equivalent either to a minimal surface in  $\mathbb{R}^3$  with  $\mathcal{K} = 1$ , or to the Clifford torus with  $\mathcal{K} = 0$ .*

The second result concerns isotropic surfaces in  $\mathbb{S}^4$  (also known as super-conformal surfaces), which is an important class of Willmore surfaces.

**Theorem B.** *Let  $y : M \rightarrow \mathbb{S}^4$  be an isotropic surface without umbilic points and  $\mathcal{K} = \text{constant}$ . Then locally  $y$  is Möbius equivalent either to a complex curve in  $\mathbb{C}^2$  with  $\mathcal{K} = 2$ , or to the Veronese sphere with  $\mathcal{K} = \frac{1}{2}$ .*

These results are remarkable in several aspects. First, Theorem A gives a local characterization of the Clifford torus in the category of Möbius geometry, which might be helpful in the exploration of Willmore Conjecture and other problems.

Next, this is the starting point to the classification of all such surfaces in  $\mathbb{S}^n$ . Surely this problem is more difficult than the original one about minimal surfaces, whereas the importance of the latter problem and its solutions have already been recognized.

Finally, although Möbius metric as well as Möbius curvature are basic invariants in the surface theory of Möbius geometry, so far they receive little attention, and there exists few results related to them (one exception is [14]). Our theorem might be the first deep result in this direction. In particular, it partly answers the following question: *To what extent is the Möbius metric restricted when we assume the surface to be Willmore?* Solution to this problem will help us to have a better understanding of Willmore surfaces. In general, one may expect to study the *intrinsic geometry* for various interesting surface classes in Möbius geometry.

Note that minimal surfaces in  $\mathbb{R}^n$  or  $\mathbb{S}^n$  are special examples of Willmore surfaces. Instead of studying such surfaces with constant Gaussian curvature, we might consider them under the condition of constant  $K_{III}$ , where  $K_{III}$  is the curvature of the metric (the third fundamental form) induced from the generalized Gauss map [10, 13, 25, 26, 27, 29]. This subject is included in our research program, because now the generalized Gauss map coincides with the conformal Gauss map in Möbius geometry, and  $K_{III} = \mathcal{K}$ . Therefore, those old results gain new meaning from our viewpoint, and provide useful informations to our study. (Note they are also related to the analogous subject: holomorphic curves in  $\mathbb{CP}^n$  with constant curvature, see [18].) In particular, as partial confirmation to the conjecture before, there are the following known results.

**Theorem.([13])** *Any minimal surface in  $\mathbb{R}^4$  with constant  $K_{III}(= \mathcal{K})$  is locally congruent to a minimal surface in  $\mathbb{R}^3$  or a complex curve in  $\mathbb{C}^2$  up to rigid motion.*

**Theorem.([25, 26])** *Any compact minimal surface in  $\mathbb{S}^4$  with constant  $K_{III}(= \mathcal{K})$  and with isolated umbilic points is congruent to the Veronese sphere in  $\mathbb{S}^4$  when it is a minimal 2-sphere, or to the Clifford torus when it is a minimal torus in  $\mathbb{S}^3$ , up to rigid motion.*

Surprisingly, all known examples of Willmore surfaces with constant  $\mathcal{K}$  are minimal surfaces in space forms. We guess that there exists no other examples. Yet it seems very hard to prove it.

Theorem A was partly obtained in [12], where the subject was studied under the name of *minimal surfaces of constant Gaussian curvature in a pseudo-Riemannian sphere*. For related work see [11]. Theorem A has also been generalized to space-like Willmore surfaces in 3-dim conformal Lorentzian geometry [23].

It is observed that the theorems above are local results except the last one. The authors believe that global assumptions are not necessary in such problems.

## 2 Surface theory in Möbius geometry

In this part we will briefly review the surfaces theory of Möbius geometry. The basic reference is [6].

As usual, the unit sphere  $\mathbb{S}^n \subset \mathbb{R}^{n+1}$  is identified with the projectivized light cone via

$$\mathbb{S}^n \cong \mathbb{P}(\mathcal{L}) : x \leftrightarrow [1, x],$$

where  $\mathcal{L} \subset \mathbb{R}^{n+1,1}$  is the null cone in the  $(n+2)$ -dim Minkowski space for the quadratic form  $\langle Y, Y \rangle = -Y_0^2 + \sum_{i=1}^{n+1} Y_i^2$ . In this way, a point in  $\mathbb{S}^n$  corresponds to an isotropic line, and a  $k$ -dim sphere is viewed as  $\mathbb{P}(\mathcal{L} \cap U^\perp)$ , which is determined by a space-like  $(n-k)$ -dim subspace  $U$  (or its orthogonal complement  $V = U^\perp$ ). The projective action of the Lorentz group on  $\mathbb{P}(\mathcal{L})$  yields all conformal diffeomorphisms of  $\mathbb{S}^n$ .

For a conformal immersion  $f : M \rightarrow \mathbb{S}^n$  of Riemann surface  $M$ , we may associate a (local) lift to it, which is just a map  $F$  from  $M$  into the light cone  $\mathcal{L}$  such that the null line spanned by  $F(p)$  is  $f(p)$ ,  $p \in M$ . Taking derivatives with respect to a given complex coordinate  $z$ , we find  $\langle F_z, F_{\bar{z}} \rangle > 0$  since  $f$  is immersed, and  $\langle F_z, F_z \rangle = 0$  since  $f$  is conformal. Furthermore, there is a rank-4 subbundle of  $M \times \mathbb{R}^{n+1,1}$ :

$$V = \text{span}\{F, \text{Re}(F_z), \text{Im}(F_z), F_{z\bar{z}},\}$$

which is independent to the choice of local lift  $F$  and complex coordinate  $z$ , hence well-defined. With signature  $(3, 1)$ ,  $V$  describes a Möbius invariant 2-sphere congruence, called the *mean curvature sphere* of  $f$ . This name comes from the property that it is tangent to the surface and has the same mean curvature as the surface at the tangent point when the ambient space is endowed with a metric of Euclidean space (or any space form).

Fix a local coordinate  $z$ . We find the canonical lift  $Y$  determined by  $|dY|^2 = |dz|^2$ . A Möbius invariant frame of  $V \otimes \mathbb{C}$  is given as

$$\{Y, Y_z, Y_{\bar{z}}, N\}, \quad (1)$$

where we choose  $N \in \Gamma(V)$  with  $\langle N, N \rangle = 0$ ,  $\langle N, Y \rangle = -1$ ,  $\langle N, Y_z \rangle = 0$ . These frame vectors are orthogonal to each other except that  $\langle Y_z, Y_{\bar{z}} \rangle = \frac{1}{2}$ ,  $\langle Y, N \rangle = -1$ .

The fundamental equation in our surface theory is

$$Y_{zz} + \frac{s}{2}Y = \kappa,$$

which holds true for some complex valued function  $s$  and a section  $\kappa \in \Gamma(V^\perp \otimes \mathbb{C})$ , because we may decompose  $Y_{zz}$  according to  $M \times \mathbb{R}^{n+1,1} = V \oplus V^\perp$ , and  $Y_{zz}$  is orthogonal to  $Y, Y_z, Y_{\bar{z}}$ . In this way we introduce two basic Möbius invariants associated with the surface  $f : M \rightarrow \mathbb{S}^n$ :  $s$  is interpreted as the Schwarzian derivative of immersion  $f$ , and  $\kappa$  may be identified with the normal-valued Hopf differential up to scaling (so  $\kappa$  vanishes exactly at the umbilic points). The *Schwarzian*  $s$  and the *Hopfian*  $\kappa$  form a complete system of conformal invariants for surfaces in  $\mathbb{S}^3$ . Later we will need the fact that  $\kappa \frac{dz^2}{|dz|}$  is a globally defined form. For more details see [6].

Let  $\xi \in \Gamma(V^\perp)$  be an arbitrary section of the normal bundle  $V^\perp$ ,  $D$  be the normal connection. We are now able to write down the structure equations:

$$\begin{cases} Y_{zz} &= -\frac{s}{2}Y + \kappa, \\ Y_{z\bar{z}} &= -\langle \kappa, \bar{\kappa} \rangle Y + \frac{1}{2}N, \\ N_z &= -2\langle \kappa, \bar{\kappa} \rangle Y_z - sY_{\bar{z}} + 2D_{\bar{z}}\kappa, \\ \xi_z &= D_z\xi + 2\langle \xi, D_{\bar{z}}\kappa \rangle Y - 2\langle \xi, \kappa \rangle Y_{\bar{z}}. \end{cases} \quad (2)$$

The conformal Gauss, Codazzi and Ricci equations as integrable conditions are

$$\frac{1}{2}s_{\bar{z}} = 3\langle D_z\bar{\kappa}, \kappa \rangle + \langle \bar{\kappa}, D_z\kappa \rangle, \quad (3)$$

$$\text{Im}(D_{\bar{z}}D_z\kappa + \frac{\bar{s}}{2}\kappa) = 0, \quad (4)$$

$$R_{\bar{z}z}^D\xi := D_{\bar{z}}D_z\xi - D_zD_{\bar{z}}\xi = 2\langle \xi, \kappa \rangle \bar{\kappa} - 2\langle \xi, \bar{\kappa} \rangle \kappa. \quad (5)$$

Since  $\kappa \frac{dz^2}{|dz|}$  is independent to the choice of coordinate  $z$ , there is a well-defined metric

$$g := e^{2\omega}|dz|^2 = 4\langle \kappa, \bar{\kappa} \rangle |dz|^2$$

over  $M$ , which is invariant under Möbius transformations and called the *Möbius metric*. Its Laplacian and Gaussian curvature  $\mathcal{K}$  (called the *Möbius curvature*) are

$$\Delta := 4e^{-2\omega}\partial_z\partial_{\bar{z}}, \quad \mathcal{K} := -\Delta\omega = -4e^{-2\omega}\omega_{z\bar{z}}.$$

It is well known that this metric is induced from the mean curvature sphere congruence and is conformal to the induced metric of  $f$ . Hence the mean curvature

sphere is also known as the *conformal Gauss map*[4]. The area of  $M$  with respect to the Möbius metric

$$W(f) := 2i \cdot \int_M |\kappa|^2 dz \wedge d\bar{z}$$

is called *Willmore functional* in this paper. It coincides with the usual definition

$$\widetilde{W}(f) := \int_M (H^2 - K) dM$$

when  $f$  is an immersed surface in  $R^3$  with mean curvature  $H$  and Gauss curvature  $K$ . The critical surfaces with respect to  $W(f)$  are called *Willmore surface*, which we are most interested in. They are characterized by the *Willmore equation* [6]:

$$D_{\bar{z}} D_{\bar{z}} \kappa + \frac{\bar{s}}{2} \kappa = 0. \quad (6)$$

Note that this is stronger than the Codazzi equation (4).

*Remark 2.1.* When the data  $\{s, \kappa\}$  satisfy (3)(6)(5), the data  $\{s, e^{it}\kappa\}$  also satisfy these integrable equations, where  $t$  is a real parameter. This yields the *associated family of Willmore surfaces*.

In the special case that  $D_{\bar{z}} \kappa$  depends linearly on  $\kappa$ , locally there is a function  $\mu$  such that

$$D_{\bar{z}} \kappa + \frac{\bar{\mu}}{2} \kappa = 0. \quad (7)$$

This time the surface is Willmore iff

$$\mu_z - \frac{1}{2} \mu^2 - s = 0. \quad (8)$$

Such a Willmore surface is called *S-Willmore surface*, which might be characterized by the property that it allows a dual Willmore surface envelopping the same mean curvature spheres [21]. Examples include Willmore surfaces in  $\mathbb{S}^3$ , isotropic surfaces in  $\mathbb{S}^4$ , and minimal surfaces in  $\mathbb{R}^n, \mathbb{S}^n, \mathbb{H}^n$ . The first surface class is S-Willmore since the codimension equals one. The second class has isotropic  $\kappa$ , whose derivative has to be linearly dependent on itself due to codim-2 (we can verify easily that it is also Willmore). Any minimal surface in a real space form has a dual surface as well. In general, the dual surface might be written down explicitly as

$$\widehat{Y} = \frac{1}{2} |\mu|^2 Y + \bar{\mu} Y_z + \mu Y_{\bar{z}} + N \quad (9)$$

with respect to the frame  $\{Y, Y_z, Y_{\bar{z}}, N\}$ . Simple calculation using (2) and (7)(8) yields

$$\widehat{Y}_{\bar{z}} = \frac{\mu}{2} \widehat{Y} + \rho \left( Y_z + \frac{\mu}{2} Y \right), \quad \text{where } \rho := \bar{\mu}_z - 2\langle \kappa, \bar{\kappa} \rangle. \quad (10)$$

Here  $\rho |dz|^2$  is a globally defined invariant associated with the surface pair  $Y, \widehat{Y}$  [21]. There follows

$$\langle \widehat{Y}_z, \widehat{Y}_z \rangle = 0, \quad \langle \widehat{Y}_z, \widehat{Y}_{\bar{z}} \rangle = \frac{1}{2} |\rho|^2. \quad (11)$$

It is straightforward to verify that  $\widehat{Y}$  share the same mean curvature sphere as  $Y$ . So  $\widehat{Y}$  determines a conformal Willmore immersion into  $\mathbb{S}^n$  when  $\rho \neq 0$  (see [21] and references therein).

**Lemma 2.2.** *For S-Willmore surfaces with  $\mu$  and  $\rho$  defined as before, we have  $\rho_{\bar{z}} = \bar{\mu}\rho$ . There exist a 4-form  $\Theta$  and a function  $\mathcal{P}$  globally defined over  $M$ :*

$$\Theta \cdot dz^4 := \rho \langle \kappa, \kappa \rangle dz^4, \quad \mathcal{P} := \frac{\rho}{\langle \kappa, \bar{\kappa} \rangle}.$$

Moreover,  $\Theta$  is holomorphic.

*Proof.* The first equality comes from (8) and (3). The holomorphicity of  $\Theta$  follows easily. Since  $\rho|dz|^2$  and  $\kappa \frac{dz^2}{|dz|}$  are globally defined, the same is true for  $\Theta$  and  $\mathcal{P}$ .  $\square$

The holomorphic 4-form  $\Theta$  was first introduced in [4] for Willmore surfaces in  $\mathbb{S}^3$ . The function  $\mathcal{P}$  will play a central role in the later discussions.

### 3 Willmore surfaces with $\mathcal{K} \equiv c$ in $\mathbb{S}^3$

In this section,  $y : M \rightarrow \mathbb{S}^3$  is always assumed to be a Willmore surface without umbilic points. Codim-1 enables us to choose a unit normal vector  $X$  canonically, which is exactly the conformal Gauss map from  $M$  into the de Sitter sphere  $\mathbb{S}^{3,1} \subset \mathbb{R}^{4,1}$ . The mean curvature sphere  $\mathbb{P}(\mathcal{L} \cap V)$  is determined by  $V = \text{span}\{Y, \text{Re}(Y_z), \text{Im}(Y_z), N, \}$  or by its orthogonal complement  $U = V^\perp = \text{span}\{X\}$ . In this way we identified the conformal Gauss map and the mean curvature sphere of  $y$ .

Now the normal-valued Hopf differential can be written as  $\kappa \cdot X$ , where we regard  $\kappa$  as a scalar function. Denote the complex valued function  $\mathcal{P}$  as

$$\mathcal{P} = |\mathcal{P}| \cdot e^{i\psi} = \frac{\rho}{\kappa \bar{\kappa}}.$$

Between scalar invariants  $\mathcal{K}$  and  $\mathcal{P}$  exists important connections as below:

$$\text{Re}(\mathcal{P}) = 2(\mathcal{K} - 1), \tag{12a}$$

$$\text{and } \Delta(\log \mathcal{P}) = 4\mathcal{K} + i \cdot \text{Im}(\mathcal{P}), \quad \text{when } |\mathcal{P}| \neq 0. \tag{12b}$$

$$\text{In particular, } \Delta(\log |\mathcal{P}|) = 4\mathcal{K}, \tag{12c}$$

$$\Delta\psi = \text{Im}(\mathcal{P}) = |\mathcal{P}| \sin \psi. \tag{12d}$$

*Proof.* From the definition of the Möbius metric and function  $\mathcal{P}$ ,

$$\mathcal{K} = -4e^{-2\omega} \omega_{z\bar{z}} = -\frac{1}{\kappa \bar{\kappa}} \left[ \frac{1}{2} \log(4\kappa \bar{\kappa}) \right]_{z\bar{z}} = \frac{1}{4\kappa \bar{\kappa}} (\bar{\mu}_z + \mu_{\bar{z}}) = \frac{1}{2} \text{Re}(\mathcal{P}) + 1$$

by (7)(10). This proves (12a), which is indeed true for any surface in  $\mathbb{S}^3$  without the Willmore condition. Next, using  $\rho_{\bar{z}} = \bar{\mu}\rho$  (Lemma 2.2), there will be

$$\begin{aligned} \Delta(\log \mathcal{P}) &= 4e^{-2\omega} [\log \rho - \log(\langle \kappa, \bar{\kappa} \rangle)]_{z\bar{z}} = 4e^{-2\omega} (\bar{\mu}_z - 2\omega_{z\bar{z}}) \\ &= \frac{1}{\kappa \bar{\kappa}} (\rho + 2\kappa \bar{\kappa}) + 2\mathcal{K} = \mathcal{P} + 2 + 2\mathcal{K}. \end{aligned}$$

Note that this is valid for any S-Willmore surface in  $\mathbb{S}^n$ . Decomposing the right hand side into real and imaginary parts and invoking (12a) yields (12b).  $\square$

The system (12a)(12b) has been obtained essentially in [24] and [20]. After posing the restriction that the metric is of constant  $\mathcal{K}$ , we obtain an over-determined system, which might be solved completely in the sequel. The first result is a well-known fact.

**Proposition 3.2.** *Let  $y : M \rightarrow \mathbb{S}^3$  be a Willmore surface. Then the following three conditions are equivalent: 1)  $\mathcal{P} \equiv 0$ ; 2)  $\mathcal{K} \equiv 1$ ; 3)  $y$  is the image of a minimal surface in  $\mathbb{R}^3$  under the inverse of a stereographic projection.*

*Proof.* We assert  $3) \Leftrightarrow 1)$ . First  $\mathcal{P} \equiv 0$  implies  $\rho \equiv 0$ , which means  $\hat{Y}$  degenerates to a single point by (11). Regarding  $y$  as in an affine space with  $\hat{Y}$  as the point at infinity, then every mean curvature sphere of  $y$  turns out to be a plane, which is of mean curvature zero. So  $y$  itself is a minimal surface in this affine  $\mathbb{R}^3$ . The converse is true for the same reason.

Note  $1) \Rightarrow 2)$  by (12a). So we need only to show  $2) \Rightarrow 1)$ . Suppose  $\mathcal{P} \neq 0$  and  $\mathcal{K} = 1$  on an open subset. Then (12a) implies  $\operatorname{Re}(F) = 0$ , hence  $\psi = \arg \mathcal{P} = \pi/2$  is a constant. It follows from (12d) that  $\mathcal{P} = 0$ , a contradiction.  $\square$

*Remark 3.3.* CMC-1 (constant mean curvature 1) surfaces in the hyperbolic space  $H^3(-1)$  also satisfy  $\mathcal{K} \equiv 1$ . Such surfaces together with their minimal cousins are characterized as the only isothermic surfaces in  $\mathbb{S}^3$  with  $\mathcal{K} \equiv 1$  [14].

**Theorem 3.4.** *Let  $y : M \rightarrow \mathbb{S}^3$  be a Willmore surface. Then the following three conditions are equivalent: 1)  $|\mathcal{P}| = \text{constant} \neq 0$ ; 2)  $\mathcal{K} \equiv 0$ ; 3)  $y$  is locally Möbius equivalent to the Clifford torus with  $\mathcal{P} = -2$ .*

*Proof.* As well-known Clifford torus is a homogeneous torus, thus  $\mathcal{K} \equiv 0$ , and  $\mathcal{P}$  is a constant. Since it is Willmore, we may use (12a)(12b) to find  $\mathcal{P} = -2$ . So  $3) \Rightarrow 1), 2)$ .

Next we prove  $1) \Rightarrow 3)$ .  $|\mathcal{P}| = \text{constant} \neq 0$  implies  $\mathcal{K} \equiv 0$  and  $\operatorname{Re}(\mathcal{P}) = -2$  by (12a)(12b). So  $\psi = \arg \mathcal{P} = \text{constant}$ . It follows from (12d) that  $\sin \psi = 0$ , hence  $\psi = 0$  and  $\operatorname{Im}(\mathcal{P}) = 0$ . We conclude that  $\mathcal{P} = -2$  and  $\mathcal{K} = 0$ . This time the Möbius metric is flat, which enables us to find a suitable coordinate  $z$  so that  $|\kappa|^2 = \frac{1}{4}e^{2\omega} = 1$ . Consequently,  $\rho = \mathcal{P} = -2$  is a constant, and  $\mu = 0$  due to  $\rho_{\bar{z}} = \bar{\mu}\rho$  (Lemma 2.2). For the basic invariants,  $s = \mu_z - \frac{1}{2}\mu^2 = 0$  by (8), and  $\kappa_{\bar{z}} = D_{\bar{z}}\kappa = 0$  by (7). Such a holomorphic and unitary function  $\kappa$  must be a constant. After a proper rotation of the coordinate  $z$  if necessary, we may assume  $\kappa = 1$  identically. So all these invariants are constant now, and the frame  $\{Y, Y_z, Y_{\bar{z}}, N, X\}$  form a linear PDE system with constant coefficients:

$$\begin{cases} Y_{zz} = X, \\ Y_{z\bar{z}} = -Y + \frac{1}{2}N, \\ N_z = -2Y_z, \\ X_z = -2Y_{\bar{z}}. \end{cases}$$

It is obviously integrable with a unique solution when the frame  $\{Y, Y_z, Y_{\bar{z}}, N, X\}$  is given at a fixed  $p \in M$ . Since the Clifford torus is a Willmore surface with  $\mathcal{K} = 0$ ,

hence also a solution to this system, it must be congruent to  $y$  up to a Möbius transformation. Indeed it may be written down explicitly at here:

$$Y(u, v) = \frac{1}{2\sqrt{2}} \cdot \left( \sqrt{2}, \cos 2\sqrt{2}u, \sin 2\sqrt{2}u, \cos 2\sqrt{2}v, \sin 2\sqrt{2}v \right).$$

Then one can verify easily that it solves the system above.

The most interesting part is to show that 2) implies 1) and 3). Suppose  $\mathcal{K} \equiv 0$ . By (12a),  $\operatorname{Re}(\mathcal{P}) = -2, \operatorname{Im}(\mathcal{P}) = \operatorname{Re}(\mathcal{P}) \cdot \tan \psi = -2 \cdot \tan \psi$ . Thus  $\mathcal{P} \neq 0$  and we may use (12d) to obtain

$$\psi_{z\bar{z}} = -\frac{1}{2}e^{2\omega} \tan \psi. \quad (13)$$

On the other hand,  $\Delta(\log |\mathcal{P}|) = 4\mathcal{K} = 0$  by (12c), so  $|\mathcal{P}|^2 = 4f\bar{f}$  for some non-zero holomorphic function  $f$ . Invoking  $|\mathcal{P}|^2 = (1 + \tan^2 \psi)\operatorname{Re}(\mathcal{P})^2$ , there is

$$\tan \psi = \sqrt{f\bar{f} - 1}, \quad \text{i.e.} \quad \psi = \arctan \sqrt{f\bar{f} - 1}.$$

Suppose  $f\bar{f} > 1$ . Direct computation shows

$$\psi_z = \frac{f_z}{f} \cdot \frac{1}{2\sqrt{f\bar{f} - 1}}, \quad \text{and} \quad \psi_{z\bar{z}} = \frac{f_z \bar{f}_{\bar{z}}}{-4(f\bar{f} - 1)^{3/2}}.$$

Substitute these expressions into (13), we find

$$e^{2\omega} = \frac{1}{2} \frac{|f_z|^2}{(1 - f\bar{f})^2}.$$

But the metric given above has constant gaussian curvature  $-8$  when  $f$  is holomorphic. A contradiction! The only possibility left is that  $f\bar{f} \equiv 1$  and  $\psi \equiv 0, \mathcal{P} \equiv -2$ . This proves 1), hence also 3).  $\square$

Theorem 3.4 gives a nice characterization of Clifford torus. Such results are always interesting, especially when they might be related to the famous Lawson's Conjecture and Willmore Conjecture. For more characterization theorems about Clifford torus, see [16, 26, 29, 30]. Below is another one using global assumptions. (The reader who is interested in general surfaces with  $\mathcal{K} = 0$  might refer to [28].)

**Theorem 3.5.** *Let  $y : M \rightarrow \mathbb{S}^3$  be a Willmore surface which is complete with respect to the Möbius metric. If  $\mathcal{K} \leq 0$  on  $M$ , then  $\mathcal{K} \equiv 0$  and  $y$  is Möbius equivalent to Clifford torus.*

*Proof.* We introduce a new metric on  $M$  as

$$\tilde{g} = \sqrt{|\mathcal{P}|}g = \sqrt{|\mathcal{P}|}e^{2\omega}|dz|^2 = e^{2\tilde{\omega}}|dz|^2, \quad (14)$$

where  $\tilde{\omega} = \frac{1}{4} \log |\mathcal{P}| + \omega$ . From the assumption  $\mathcal{K} \leq 0$  and (12d), we have  $\operatorname{Re}(\mathcal{P}) \leq -2$ , hence  $|\mathcal{P}| \geq 2$ . So the new metric is positive definite and complete, too. The Gaussian curvature of  $(M, \tilde{g})$  is

$$\tilde{K} = -4e^{-2\tilde{\omega}}(\tilde{\omega})_{z\bar{z}} = 0,$$



by (12c). We may assume that  $M$  is simply connected. Therefore  $(M, \tilde{g})$  is an Euclidean plane, and  $(M, g)$  is conformally equivalent to the complex plane  $\mathbb{C}$ . Since  $|\mathcal{P}| \geq 2$  and  $\mathcal{K} \leq 0$ , by (12c),  $\log |\mathcal{P}|$  is a subharmonic function on  $(\mathbb{C}, \tilde{g})$  with lower bound  $\log 2$ . We conclude that  $|\mathcal{P}| = \text{constant} > 0$ . The proof is finished by using Theorem 3.4.  $\square$

Now we can prove the first main theorem.

**Theorem 3.6.** *Let  $y : M \rightarrow \mathbb{S}^3$  be a Willmore surface with constant Möbius curvature  $\mathcal{K}$ . Then  $y$  is Möbius equivalent either to a minimal surface in  $\mathbb{R}^3$  with  $\mathcal{K} = 1, \mathcal{P} = 0$ , or to the Clifford torus with  $\mathcal{K} = 0, \mathcal{P} = -2$ .*

*Proof.* Let's assume  $\mathcal{K} = \text{constant} \neq 0, 1$ . This time  $\psi$  could not be a constant. (Otherwise it contradicts the assumption by (12a)(12b).) The first step is to show that this real function  $\psi : M \rightarrow \mathbb{R}$  is an isoparametric function.

First,  $\text{Re}(\mathcal{P}) = 2(\mathcal{K} - 1)$  is a non-zero constant. Note

$$\text{Im}(\mathcal{P}) = \text{Re}(\mathcal{P}) \cdot \tan \psi = 2(\mathcal{K} - 1) \tan \psi, \quad (15)$$

$$|\mathcal{P}| = \text{Re}(\mathcal{P}) / \cos \psi = 2(\mathcal{K} - 1) \sec \psi. \quad (16)$$

Substitute (15) and (16) separately into  $\Delta \psi = \text{Im}(\mathcal{P})$  (12d) and  $\Delta \log |\mathcal{P}| = 4\mathcal{K}$  (12c), we find

$$\Delta \psi = 2(\mathcal{K} - 1) \tan \psi = G(\psi), \quad (17)$$

$$\|\nabla \psi\|^2 = 4e^{-2\omega} |\psi_z|^2 = 4 \left( \mathcal{K} \cos^2 \psi - \frac{\mathcal{K} - 1}{2} \sin^2 \psi \right) = F(\psi). \quad (18)$$

Here (18) follows from (12c)(17) via the following computation:

$$\begin{aligned} e^{2\omega} K &= (\log |\mathcal{P}|)_{z\bar{z}} = (-\log \cos \psi)_{z\bar{z}} = \psi_{z\bar{z}} \cdot \tan \psi + \psi_z \psi_{\bar{z}} \sec^2 \psi \\ &= \frac{\mathcal{K} - 1}{2} e^{2\omega} \tan^2 \psi + \psi_z \psi_{\bar{z}} \sec^2 \psi. \end{aligned}$$

Such a function  $\psi$  satisfying  $\Delta \psi = G(\psi)$ ,  $\|\nabla \psi\|^2 = F(\psi)$  is called an *isoparametric function*, where  $F, G$  are two functions of  $\psi$ .

We want to find contradiction in these equations. This is done by appealing to the following identity (see Lemma 5.1):

$$2\mathcal{K}F + (2G - F')(G - F') + F(2G' - F'') = 0, \quad (19)$$

where  $G' = dG/d\psi$ ,  $F' = dF/d\psi$ , etc.. Substituting the expressions of  $F(\psi)$  and  $G(\psi)$  (17)(18) into this identity, after simplification we find

$$4(27\mathcal{K} - 8)(\mathcal{K} - 1) - 4(3\mathcal{K} - 1)(3\mathcal{K} - 8) \cos^2 \psi = 0.$$

The left hand side is a polynomial of  $\cos \psi$ . It is identically zero iff all the coefficients vanish. Yet this is obviously impossible for any constant  $\mathcal{K}$ . A contradiction! Thus our assumption in the beginning is wrong. The conclusion follows from Proposition 3.2 and Theorem 3.4.  $\square$

*Remark 3.7.* Gorokh[12] studied minimal surfaces of constant Gaussian curvature in the 4-dim de Sitter's space  $\mathbb{S}^{3,1}(1)$ . Note that such surfaces are equivalent to Willmore surfaces of constant Möbius curvature in  $\mathbb{S}^3$ . Any surface of the former class, denoted as  $X$ , is exactly the conformal Gauss map of one in the latter class, and vice versa. Except the case  $\mathcal{K} = 1$  where  $X$  is totally umbilic, he obtained the same result as Theorem 3.6. (He announced that the exceptional case  $\mathcal{K} = 1$ , also the easiest case, would be indicated in his next paper, which did not appear.) His method is similar to our previous proof in essence, yet buried in tedious computations. We think that the same result in terms of Willmore surfaces might interest more people, and our presentation not only clarifies the key points, but also illustrates a general method for solving such problems.

As a by-product of (12a)(12b), we find the following characterization of the associated Willmore surfaces.

**Proposition 3.8.** *Let  $y, \tilde{y} : M \rightarrow \mathbb{S}^3$  be two conformal Willmore immersions. The invariants of  $y$  satisfy  $\mathcal{P} \neq 0$  and  $\mathcal{K} \neq 0$ . Then  $\mathcal{P} = \tilde{\mathcal{P}}$  iff  $y$  and  $\tilde{y}$  are in the same associated family.*

*Proof.* When  $y$  and  $\tilde{y}$  are in the same associated family, their Hopfians are related by  $\tilde{\kappa} = e^{it}\kappa$  for a constant  $t \in \mathbb{R}$ . Thus they share the same  $\mu$  and  $\langle \kappa, \bar{\kappa} \rangle$  (Möbius metric). By definition  $\mathcal{P} = \tilde{\mathcal{P}}$ . So we need only to show the converse.

Choose the same coordinate  $z$  for  $y$  and  $\tilde{y}$ . Then it follows from (12a) that  $\mathcal{K} = \tilde{\mathcal{K}}$ . By (12c),  $(e^{-2\omega} - e^{-2\tilde{\omega}})(\log |\mathcal{P}|)_{z\bar{z}} = 0$ . Because  $\mathcal{K} \neq 0$ ,  $(\log |\mathcal{P}|)_{z\bar{z}} \neq 0$ , so  $\omega = \tilde{\omega}$ , i.e.  $x, \tilde{x}$  induce the same Möbius metric. Combined with  $\mathcal{P} = \tilde{\mathcal{P}} \neq 0$ , we have  $\rho = \tilde{\rho} \neq 0$ . Taking logarithm at both sides and differentiating yields  $\mu = \tilde{\mu}$  by Lemma 2.2. As the consequence, their Schwarzians are equal due to (8), and their Hopfians differ by a constant factor, i.e.  $\tilde{\kappa} = e^{it}\kappa$ , due to (7). So  $y, \tilde{y}$  are in the same associated family of Willmore surfaces.  $\square$

## 4 Isotropic surfaces with $\mathcal{K} \equiv c$ in $\mathbb{S}^4$

A surface in  $\mathbb{S}^4$  is isotropic if  $\langle \kappa, \kappa \rangle = 0$ . This is a Möbius invariant notion, although usually defined in terms of metric geometry. For its important geometric meaning we refer to [5, Ch. 8].

The isotropic condition implies that all derivatives of the Hopfian,  $\{D_z\kappa, D_{\bar{z}}\kappa, \dots\}$ , are contained in the isotropic line bundle  $\text{span}\{\kappa\}$ . As a corollary of this fact and the conformal Codazzi equation (4), such surfaces must be S-Willmore.

From now on we always assume that  $y : M \rightarrow \mathbb{S}^4$  is an isotropic surface without umbilic points. So  $\kappa \neq 0$  and we have  $D_{\bar{z}}\kappa = -\frac{\bar{\mu}}{2}\kappa$ ,  $D_z\kappa = \lambda\kappa$  for some locally defined function  $\mu, \lambda$ . Substitute them into the conformal Ricci equation (5)

$$D_{\bar{z}}D_z\kappa = D_zD_{\bar{z}}\kappa - 2\langle \kappa, \bar{\kappa} \rangle \kappa + 2\langle \kappa, \kappa \rangle \bar{\kappa}.$$

It follows  $\bar{\lambda}_z = -\frac{\mu\bar{z}}{2} - 2\langle\kappa, \bar{\kappa}\rangle$ . Then the Möbius curvature is computed as below:

$$\begin{aligned}\mathcal{K} &= -\frac{1}{2\langle\kappa, \bar{\kappa}\rangle}(\log\langle\kappa, \bar{\kappa}\rangle)_{z\bar{z}} = -\frac{1}{2\langle\kappa, \bar{\kappa}\rangle}\left(-\frac{\bar{\mu}}{2} + \bar{\lambda}\right)_z \\ &= \frac{1}{4\langle\kappa, \bar{\kappa}\rangle}(\rho + \bar{\rho} + 8\langle\kappa, \bar{\kappa}\rangle) = \frac{1}{2}\text{Re}(\mathcal{P}) + 2,\end{aligned}$$

where  $\rho$  and  $\mathcal{P}$  are defined as before. On the other hand, in the proof to Lemma 3.1, it was noticed that for any S-Willmore surface in  $\mathbb{S}^n$  holds  $\Delta(\log \mathcal{P}) = \mathcal{P} + 2 + 2\mathcal{K}$  if  $\mathcal{P} \neq 0$ . Combined with the previous formula, we have proved

**Lemma 4.1.** *For any isotropic surface satisfying  $\mathcal{P} \neq 0$ , there are*

$$\mathcal{K} = \frac{1}{2}\text{Re}(\mathcal{P}) + 2 = \frac{1}{2}|\mathcal{P}| \cos \psi + 2, \quad (20a)$$

$$\Delta(\log |\mathcal{P}|) = 4\mathcal{K} - 2, \quad (20b)$$

$$\Delta\psi = \text{Im}(\mathcal{P}) = |\mathcal{P}| \sin \psi, \quad (20c)$$

Where  $\psi$  is the argument of  $\mathcal{P}$ . Similar as in the previous section, with the help of this lemma we prove two characterization theorems, then complete the final classification.

**Theorem 4.2.** *Let  $y : M \rightarrow \mathbb{S}^4$  be an isotropic surface without umbilic points. Then the following three conditions are equivalent: 1)  $\mathcal{P} \equiv 0$ ; 2)  $\mathcal{K} \equiv 2$ ; 3) Locally  $y$  is Möbius equivalent to a complex curve in  $\mathbb{C}^2 \cong \mathbb{R}^4$ , where the identification is made by choosing a suitable orthogonal complex structure on  $\mathbb{R}^4$ .*

*Proof.* If  $\mathcal{P} = 0$  on an open subset of  $M$ . Then the dual Willmore surface  $\hat{Y}$  will degenerate to a single point as we observed before. This case  $y$  is Möbius equivalent to a minimal surface in an affine 4-space. It is known that an isotropic minimal surface in  $\mathbb{R}^4$  is a complex curve in  $\mathbb{R}^4 \cong \mathbb{C}^2$ . (See [13] for a proof. The straightforward way is to define a complex structure  $J$  on  $TM$  and  $T^\perp M$ , then show its parallel translation  $\nabla J = 0$ . Hence it extends to a complex structure on  $\mathbb{R}^4$ .) By (20a),  $\mathcal{K} \equiv 2$ . Conversely,  $\mathcal{K} \equiv 2$  implies  $\mathcal{P} \equiv 0$  like in Proposition 3.2. So complex curves in  $\mathbb{C}^2$  are isotropic, minimal and  $\mathcal{K} = 2$ , and they are characterized by the combination of any two of these conditions.  $\square$

**Theorem 4.3.** *Let  $y : M \rightarrow \mathbb{S}^4$  be an isotropic surface without umbilic points. Then the following three conditions are equivalent: 1)  $|\mathcal{P}| = \text{constant} \neq 0$ ; 2)  $\mathcal{K} \equiv \frac{1}{2}$ ; 3) Locally  $y$  is Möbius equivalent to the Veronese sphere in  $\mathbb{S}^4$  with  $\mathcal{P} \equiv -3$ .*

*Proof.* The Veronese 2-sphere is also a homogeneous Willmore surface, thus  $\mathcal{K}, \mathcal{P}$  are both constant, which could be found as  $\mathcal{K} = \frac{1}{2}, \mathcal{P} = -2$  by (20a)-(20c). (The case  $\mathcal{P} = 0$  is impossible because we know that the Veronese surface is only minimal in  $\mathbb{S}^4$  and not minimal in any Euclidean space. Indeed  $\mathcal{K} = \frac{1}{2}$  was a known fact about this surface.) So 3) $\Rightarrow$ 1), 2).

Next we prove 1) $\Rightarrow$ 2), 3).  $|\mathcal{P}| = \text{constant} \neq 0$  implies  $\mathcal{K} = \frac{1}{2}$  and  $\text{Re}(\mathcal{P}) = -3$  by (20b)(20a). So the argument  $\psi$  is constant. It follows from (20c) that  $\sin \psi = 0$ ,

hence  $\psi = 0$  and  $\text{Im}(\mathcal{P}) = 0, \mathcal{P} = -3$ . By the definitions of  $\mathcal{P}$  and Lemma 2.2, one finds

$$\rho = -3\langle\kappa, \bar{\kappa}\rangle, \quad \bar{\mu} = (\log \rho)_{\bar{z}} = (\log \langle\kappa, \bar{\kappa}\rangle)_{\bar{z}} = 2\omega_{\bar{z}}.$$

Together with  $(\log \langle\kappa, \bar{\kappa}\rangle)_{\bar{z}} = -\frac{\bar{\mu}}{2} + \bar{\lambda}$  there is  $\bar{\lambda} = 3\omega_{\bar{z}}$ , and the Schwarzian is

$$s = \mu_z - \frac{1}{2}\mu^2 = 2(\omega_{zz} - \omega_z^2),$$

which is holomorphic by (3) or by  $\omega_{z\bar{z}} = -\frac{1}{8}e^{2\omega}$ . Sum together, we have shown that the complex frame  $\{Y, Y_z, N, \kappa\}$  of this isotropic surface satisfies

$$\begin{cases} Y_{zz} = (\omega_z^2 - \omega_{zz})Y + \kappa, \\ Y_{z\bar{z}} = -\frac{1}{4}e^{2\omega} Y + \frac{1}{2}N, \\ N_z = -\frac{1}{2}e^{2\omega} Y_z + 2(\omega_z^2 - \omega_{zz})Y_{\bar{z}} - 2\omega_{\bar{z}}\kappa, \\ \kappa_z = D_z\kappa = 3\omega_z\kappa, \\ \kappa_{\bar{z}} = D_{\bar{z}}\kappa + 2\langle\kappa, D_z\bar{\kappa}\rangle Y - 2\langle\kappa, \bar{\kappa}\rangle Y_z \\ \quad = -\omega_{\bar{z}}\kappa - \frac{1}{2}\omega_z e^{2\omega} Y - \frac{1}{2}e^{2\omega} Y_z \end{cases}$$

with respect to a given coordinate  $z$  and a metric  $e^{2\omega}|dz|^2$  of constant curvature  $\mathcal{K} = \frac{1}{2}$ . This PDE system is integrable by our construction and verification. The solution is unique when the initial values of these frame vectors at a given  $p \in M$  are fixed. So  $Y$  is locally congruent to the known solution, the Veronese sphere, up to a Möbius transformation.

The final part “2) $\Rightarrow$ 1),3)” will be included in the proof to the next theorem.  $\square$

**Theorem 4.4.** *Let  $y : M \rightarrow \mathbb{S}^4$  be an isotropic surface without umbilic points and  $\mathcal{K} = \text{constant}$ . Then locally  $y$  is Möbius equivalent either to a complex curve in  $\mathbb{C}^2$  with  $\mathcal{K} = 2, \mathcal{P} = 0$ , or to the Veronese sphere with  $\mathcal{K} = \frac{1}{2}, \mathcal{P} = -3$ .*

*Proof.* Suppose the constant  $\mathcal{K} \neq 2$ , or equivalently  $\mathcal{P} \neq 0$ , on an open subset of  $M$ . Then  $\text{Re}(\mathcal{P}) = 2\mathcal{K} - 4$  is a non-zero constant. Put  $\text{Im}(\mathcal{P}) = 2(\mathcal{K} - 2) \tan \psi$  and  $|\mathcal{P}| = 2(\mathcal{K} - 2) \sec \psi$  into (20c)(20b), we find

$$\Delta\psi = 2(\mathcal{K} - 2) \tan \psi = G(\psi), \quad (21)$$

$$\|\nabla\psi\|^2 = 4 \left( \mathcal{K} \cos^2 \psi - \frac{\mathcal{K} - 2}{2} \sin^2 \psi \right) = F(\psi) \quad (22)$$

just as we did in the proof to Theorem 3.6. Substitute the expressions of  $F(\psi)$  and  $G(\psi)$  (21)(22) into the same identity (19) for isoparametric function  $\psi$ :

$$2\mathcal{K}F + (2G - F')(G - F') + F(2G' - F'') = 0.$$

We conclude in the same manner that when  $\psi$  is a non-constant function there exists no such a  $\mathcal{K}$  satisfying this identity. The left possibility is  $\psi = \text{constant}$ . (In particular, the condition “ $\mathcal{K} \equiv \frac{1}{2}$ ” in Theorem 4.3 implies  $\psi = \text{constant}$ .) This constant  $\psi$  has to be zero due to (20c), hence  $\mathcal{P} = \text{Re}(\mathcal{P})$  is also constant by (20a), and  $\mathcal{K} \equiv \frac{1}{2}$ . This surface is Möbius equivalent to the Veronese 2-sphere by the “1) $\Rightarrow$ 3)” part in Theorem 4.3. Thus we complete the proof to Theorem 4.3 and Theorem 4.4 at the same time.  $\square$

*Remark 4.5.* Like that for Clifford torus, characterization of Veronese surface(s) is also a favorite topic in the study of submanifolds with abundant results. Here we would like to emphasize that our characterizations of Clifford torus and Veronese 2-sphere are in terms of conformal invariants. Another such result is [19].

## 5 Further remarks

We note that our results are good examples of the phenomenon of *quantization* and *non-negativity* in such problems. Namely, certain geometric quantity (like the scalar curvature) of those rigid objects (like minimal submanifolds), when being constant, can take its value only in a discrete subset of  $\mathbb{R}$ . In particular, the negative values are usually forbidden, like minimal surfaces in  $\mathbb{S}^n$  of constant Gaussian curvature mentioned at the beginning.

The equation systems we met in Lemma 3.1 and Lemma 4.1 are examples of Toda system, an important class of integrable systems. They are usually associated with certain surface classes: besides Willmore surfaces in  $\mathbb{S}^3$  and isotropic surfaces in  $\mathbb{S}^4$ , there are Willmore surfaces in  $\mathbb{S}^{2,1}$  [1], minimal surfaces in  $\mathbb{S}^4$  [16] and (non-super) minimal surfaces in  $\mathbb{CP}^2$  [8] among others.

In such systems, under the assumption of constant Gaussian curvature of the metric in consideration, one usually find some invariants being isoparametric functions. For results concerning such functions the reader might consult [8] and references therein. The most useful property of such functions on a Riemannian surface is the identity (19). Many people used it in their work [8, 12, 16, 17, 26, 29, 30]. It is usually referred to Eisenhart's book [9, p.164]. Yet this identity is not given explicitly there. Here we state it as the lemma below with an independent proof.

**Lemma 5.1.** *Let  $(M, ds^2)$  be a 2-dim Riemannian manifold and  $\psi : M \rightarrow \mathbb{R}$  be a smooth isoparametric function, i.e.*

$$||\nabla\psi||^2 = F(\psi), \quad \Delta\psi = G(\psi)$$

*for smooth functions  $F, G : \mathbb{R} \rightarrow \mathbb{R}$ , where  $\nabla$  and  $\Delta$  denote the gradient and Laplacian with respect to the metric  $ds^2$ . Then on  $\{p \in M : \nabla\psi(p) \neq 0\}$ , the Gaussian curvature  $\mathcal{K}$  satisfies the identity*

$$2\mathcal{K}F + (2G - F')(G - F') + F(2G' - F'') = 0.$$

*Here  $G' = dG/d\psi, F' = dF/d\psi$ , etc..*

*Proof.* Write  $ds^2 = e^{2\omega}|dz|^2$ . Then  $F(\psi) = 4e^{-2\omega}\psi_z\psi_{\bar{z}}, G(\psi) = 4e^{-2\omega}\psi_{z\bar{z}}$ . Eliminating  $e^{2\omega}$ , we find

$$\psi_{z\bar{z}} = f_z f_{\bar{z}} \cdot \frac{G}{F}. \quad (23)$$

On the other hand, there follows  $e^{2\omega} = |\frac{2\psi_z}{\sqrt{F}}|^2$ . To compute its Gaussian curvature,

we notice that

$$\begin{aligned} \left( \frac{\psi_z}{\sqrt{F}} \right)_{\bar{z}} &= \frac{\psi_{z\bar{z}}}{\sqrt{F}} - \frac{1}{2} \frac{\psi_z}{F\sqrt{F}} F_{\bar{z}} = \frac{\psi_z \psi_{\bar{z}}}{F\sqrt{F}} G - \frac{1}{2} \frac{\psi_z}{F\sqrt{F}} \psi_{\bar{z}} \cdot F' \\ &= \frac{\psi_z}{\sqrt{F}} \psi_{\bar{z}} \cdot \frac{df}{d\psi} = \frac{\psi_z}{\sqrt{F}} \cdot f_{\bar{z}} \end{aligned}$$

by (23), where  $f(\psi)$  is a real function solving the ODE  $f' = \frac{1}{F} (G - \frac{1}{2}F')$ . There follows  $(\log \frac{2\psi_z}{\sqrt{F}} - f)_{\bar{z}} = 0$ , hence  $\frac{2\psi_z}{\sqrt{F}} = e^f \cdot h$  for some holomorphic function  $h$  on  $M$ . Write  $e^{2\omega} = e^{2f} \cdot |h|^2$ . The curvature  $\mathcal{K}$  satisfies

$$\begin{aligned} -\mathcal{K} &= 4e^{-2\omega} \omega_{z\bar{z}} = 4e^{-2\omega} f_{z\bar{z}} = 4e^{-2\omega} \left[ \frac{\psi_{\bar{z}}}{F} (G - \frac{1}{2}F') \right]_z \\ &= 4e^{-2\omega} \left[ \left( \frac{\psi_{z\bar{z}}}{F} - \frac{\psi_z \psi_{\bar{z}}}{F^2} \cdot F' \right) (G - \frac{1}{2}F') + \frac{\psi_{\bar{z}}}{F} \cdot \psi_z \cdot (G' - \frac{1}{2}F'') \right] \\ &= \frac{1}{F} \left[ (G - F')(G - \frac{1}{2}F') + F(G' - \frac{1}{2}F'') \right]. \quad \square \end{aligned}$$

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